On-shell supercurrent multiplet for supergravity in six dimensions

1981 J. Phys. A: Math. Gen. 14 1977

(http://iopscience.iop.org/0305-4470/14/8/022)

J. Phys. A: Math. Gen. **14** (1981) 1977–1981. Printed in Great Britain

On-shell supercurrent multiplet for supergravity in six dimensions†

R Puzalowski

Department of Physics, Brandeis University, Waltham, MA 02254, USA

Received 1 September 1980, in final form 10 December 1980

Abstract. We use unextended on-shell superfields to construct the Noether current multiplet for supergravity in six dimensions. The resulting current superfield is shown to generate all exotic currents (besides the canonical ones). This procedure therefore provides a systematic way to determine them. The invariant which reduces to the three-loop counterterm of O(2) supergravity is given in terms of the current superfield.

1. Introduction

It has been shown (Deser and Lindström 1980) that it is possible to construct a supersymmetry invariant in six-dimensional space–time as a functional of the unextended matter fields λ and $F_{\mu\nu}$ which reduces to the matter part of the three-loop counterterm for O(2) supergravity via dimensional reduction (Scherk 1979). It was also demonstrated that this procedure of finding counterterms is much more convenient than a 4D approach, since one has fewer currents to deal with (Deser and Kay 1978, Sohnius 1979). However, the construction showed also that exotic currents may appear in higher dimensions, that is, currents which do not have 4D counterparts. Since it is not obvious from the beginning what the exotic currents look like, the question arises whether there is a systematic way to find these additional currents and how the current multiplet generalises.

In order to answer this question, at least for the next more complicated case, the gravitational part of the three-loop counterterm, we use a formulation in terms of 6D superfields (Siegel 1979). This approach automatically guarantees that we obtain all members of the current multiplet. In performing this program we find, as a result, that the exotic matter current $D_{\mu\nu}$ defined by Deser and Lindström (1980) in fact appears as a component field of the current superfield, together with another exotic current which is new. We find an analogous structure of the current multiplet for the gravitational part. However, in constructing the invariant in question, it turns out that the matter contribution is reducible due to the equations of motion, and we end up with the same expression as Deser and Lindström (1980).

† Work supported in part by German Academic Exchange Service (DAAD) and NSF Grant No PHY-78-09644.

0305-4470/81/081977+05\$01.50

2. Construction

We begin our construction with the matter part. On shell† the supermultiplet containing a spinor λ and vector field $F_{\mu\nu}$ can be represented by a 6D superfield W^A‡,

$$W^A = \lambda^A - i\bar{\theta}\sigma^\mu\theta\, \partial_\mu\lambda^A - i\theta\sigma^\mu\theta\, \partial_\mu\bar{\lambda}^A + \tfrac{1}{4}\bar{\theta}\sigma^\mu\bar{\theta}\theta\sigma^\nu\theta\, \partial_\mu\, \partial_\nu\lambda^A$$
$$+\tfrac{1}{2}i(\sigma^{\mu\nu}\theta)^A F_{\mu\nu} + \tfrac{1}{4}\theta\sigma^\lambda\theta(\bar{\theta}\sigma^{\mu\nu})^A\, \partial_\lambda F_{\mu\nu} \tag{1}$$

with the supertransformations

$$\delta\lambda^A = -\tfrac{1}{2}(\sigma^{\mu\nu}\alpha)^A F_{\mu\nu},$$
$$\delta F_{\mu\nu} = i(\bar{\alpha}\sigma_{[\mu}\, \partial_{\nu]}\lambda - \partial_{[\mu}\bar{\lambda}\sigma_{\nu]}\alpha).$$

We found it very convenient to employ a 6D Weyl representation for the spinors (Akyeampong and Delbourgo 1973, Siegel 1979)§. The generalised 4×4 Pauli matrices σ_{AB} here represent only half of the total eight-dimensional space, which is sufficient however, since the corresponding Dirac spinors in this space appear only as helicity projections (Deser and Lindström 1980, Siegel 1979). The following properties of the σ's are useful:

$$\{\sigma^\alpha, \sigma^\beta\} = 2\eta^{\alpha\beta}, \qquad \sigma^{\alpha AB} = \tfrac{1}{2}\varepsilon^{ABCD}\sigma^\alpha{}_{CD},$$
$$\sigma^\alpha{}_{AB}\sigma_{\alpha CD} = 2\varepsilon_{ABCD},$$

and we define

$$\sigma^{\alpha\beta} = \tfrac{1}{2}\sigma^{[\alpha}\sigma^{\beta]}.$$

In this notation the covariant derivatives D_A, $\bar{D}_B$ and D_α obey the algebra

$$\{D_A, \bar{D}_B\} = -2\sigma^\alpha{}_{AB}D_\alpha$$

and the field W^A is a solution to

$$\sigma^\alpha{}_{AB}[D_\alpha, W^B] = 0, \qquad \{\bar{D}_A, W^B\} = 0,$$
$$\{D_A, W^B\} = \{\bar{D}_A, \bar{W}^B\}, \qquad [\bar{D}_A, \{D_B, W^C\}] = 2\sigma^\alpha{}_{AB}[D_\alpha, W^C], \tag{2}$$
$$\{D_A, [D_B, \{D_C, W^D\}]\} = 0, \qquad \sigma^{\alpha AB}[D_\alpha, \{D_B, W^C\}] = 0.$$

In the weak field limit (Deser and Kay 1978) we may also represent the gravitational multiplet by a 6D superfield $W^A{}_{\alpha\beta} = -W^A{}_{\beta\alpha}$ which we obtain from (1) by substituting $\lambda^A \to f^A{}_{\alpha\beta}$, the '$\frac{3}{2}$ field strength', and $F_{\mu\nu} \to R_{\alpha\beta\mu\nu}$, the 'Riemann tensor', with the supertransformations

$$\delta f_{\alpha\beta} = -\tfrac{1}{2}\sigma^{\mu\nu}\alpha R_{\alpha\beta\mu\nu}, \qquad \delta R^{\alpha\beta}{}_{\mu\nu} = i(\bar{\alpha}\sigma_{[\mu}\, \partial_{\nu]}f^{\alpha\beta} - \partial_{[\mu}\bar{f}^{\alpha\beta}\sigma_{\nu]}\alpha).$$

We construct the current superfield in terms of the W fields as in the four-dimensional case (Ferrara and Zumino 1975), that is, the current multiplets are given by the real superfields V^μ and $V^\mu{}_{\alpha\beta}$ defined by

$$V^\mu = \bar{W}\sigma^\mu W, \qquad V^\mu{}_{\alpha\beta} = \bar{W}_{(\alpha}{}^\rho\sigma^\mu W_{\beta)\rho}$$

† In what follows all statements are modulo equations of motion, and we work in the weak field limit as defined in the work by Deser and Lindström (1980) and Deser and Kay (1978).

‡ Index convention: $A, B, C, \ldots$ spinor indices, $\alpha, \beta, \gamma, \ldots$ 6D local indices, $a, b, c, \ldots$ 4D local indices, $i, j, k, \ldots$ internal indices.

§ We use the flat space metric with positive signature and $\varepsilon_{012345} = +1$.

for the matter and gravitational part respectively. These currents are conserved as a consequence of (2).

We give here explicitly the components of the gravitational multiplet defined in terms of the commutators of $V^{\mu}{}_{\alpha\beta}$ with the covariant derivatives at $\theta=\bar{\theta}=0$:

$$
\begin{aligned}
&V^{\mu}{}_{\alpha\beta}(x,0)\equiv C^{\mu}{}_{\alpha\beta}(x), \qquad [D_A, V^{\mu}{}_{\alpha\beta}(x,0)]\equiv J^{\mu}{}_{\alpha\beta A}(x),\\
&\tfrac{1}{8}\sigma^{\lambda AB}\{\bar{D}_A,[\bar{D}_B, V^{\mu}{}_{\alpha\beta}(x,0)]\}\equiv t^{\lambda\mu}{}_{\alpha\beta}(x),\\
&\tfrac{1}{8}(\sigma^{\lambda AB}\{D_A,[\bar{D}_B, V^{\mu}{}_{\alpha\beta}(x,0)]\}+\mathrm{HC})\equiv T^{\lambda\mu}{}_{\alpha\beta}(x)+D^{\lambda\mu}{}_{\alpha\beta}(x),
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
t^{\lambda\mu}{}_{\alpha\beta}&=\mathrm{i}\,\partial^{\lambda}f^{\rho}{}_{(\alpha}\sigma^{\mu}f_{\beta)\rho},\\
T^{\lambda\mu}{}_{\alpha\beta}&=-\mathrm{i}(\partial^{\lambda}\bar{f}_{(\alpha}{}^{\rho}\sigma^{\mu}f_{\beta)\rho}-\bar{f}_{(\alpha}{}^{\rho}\sigma^{\mu}\,\partial^{\lambda}f_{\beta)\rho})+R_{(\alpha}{}^{\nu\rho\mu}R_{\beta)\nu\rho}{}^{\lambda}+(1/3!)R^{*}_{(\alpha}{}^{\nu\rho\sigma\tau\mu}R^{*}_{\beta)\nu\rho\sigma\tau}{}^{\lambda},\\
D^{\lambda\mu}{}_{\alpha\beta}&=\tfrac{1}{2}R^{*}{}_{(\alpha}{}^{\nu\rho\sigma\lambda\mu}R_{\beta)\nu\rho\sigma}, \qquad R^{*}{}_{\alpha\beta}{}^{\lambda\mu\nu\rho}\equiv\tfrac{1}{2}\varepsilon^{\lambda\mu\nu\rho\sigma\tau}R_{\alpha\beta\sigma\tau}.
\end{aligned}
$$

$C^{\mu}{}_{\alpha\beta}$ and $J^{\mu}{}_{\alpha\beta}$ are the axial current and supercurrent respectively. $T^{\lambda\mu}{}_{\alpha\beta}$ contains the Bel–Robinson tensor. The currents $t^{\lambda\mu}{}_{\alpha\beta}$ and $D^{\lambda\mu}{}_{\alpha\beta}$ are exotic in the sense above. $D^{\lambda\mu}{}_{\alpha\beta}$ in particular corresponds to $D^{\lambda\mu}$ defined by Deser and Lindström (1980) for the matter part. All other components of $V^{\mu}{}_{\alpha\beta}$ can now be obtained from these, since all higher derivatives of $V^{\mu}{}_{\alpha\beta}$ can be reduced to those already known by means of (2). We have for instance

$$
(1/3!)\varepsilon^{ABCD}[\bar{D}_A,\{\bar{D}_B,[D_C, V^{\mu}{}_{\alpha\beta}]\}]=\sigma^{\lambda CD}[D_\lambda,[\bar{D}_C, V^{\mu}{}_{\alpha\beta}]]
$$

or

$$
\{D_A,[D_B,\{\bar{D}_C,[\bar{D}_D, V^{\mu}{}_{\alpha\beta}]\}]\}=\sigma^{\lambda}{}_{CD}[D_\lambda,\{D_{[A},[\bar{D}_{B]}, V^{\mu}{}_{\alpha\beta}]\}]
$$

etc. The matter part is completely analogous.

In order to construct an invariant I_{grav}, which has after dimensional reduction the dimension of a three-loop counterterm containing in particular the square of the Bel–Robinson tensor, we have to square $V^{\mu}{}_{\alpha\beta}$. However, it is clear from dimensional considerations as well as from the results above, that I_{grav} does not occur as a last component of the superfield V^2, but will be in general a collection of components of the θ-basis elements which are quadratic in θ. We follow again the procedure described by Siegel (1979), and construct I_{grav} by projecting the desired components of V^2 down to the first component of the superfield which consists of derivatives of $V^{\mu}{}_{\alpha\beta}$, that is terms like $I_{\mathrm{grav}}\sim(D_AD_BV^{\mu}{}_{\alpha\beta})^2$. In order to be an invariant, $I_{\mathrm{grav}}=\int \mathrm{d}^6xL$ should have the property $\partial(I_{\mathrm{grav}})/\partial\theta^A=0$, which implies $[D_A, L]=0$ and $[Q_A, L]=0$.

Using the algebra (2), we find I_{grav} has the form

$$
I_{\mathrm{grav}}=\int \mathrm{d}^6x(L_1+L_2+\tfrac{2}{3}L_3-16L_4)
$$

where

$$
\begin{aligned}
L_1&=\varepsilon^{ABCD}\{\bar{D}_A,[\bar{D}_B, V^{\mu}{}_{\alpha\beta}]\}\{D_C,[D_D, V_{\mu}{}^{\alpha\beta}]\},\\
L_2&=\varepsilon^{ABCD}\{\bar{D}_A,[D_B, V^{\mu}{}_{\alpha\beta}]\}\{D_C,[\bar{D}_D, V_{\mu}{}^{\alpha\beta}]\},\\
L_3&=(\varepsilon^{ABCD}[\bar{D}_A,\{\bar{D}_B,[D_C, V^{\mu}{}_{\alpha\beta}]\}][D_D, V_{\mu}{}^{\alpha\beta}]+\mathrm{HC}),\\
L_4&=[D^{\lambda}, V^{\mu}{}_{\alpha\beta}][D_\lambda, V_{\mu}{}^{\alpha\beta}].
\end{aligned}
\tag{4}
$$

For the matter part, I_{matter} can be reduced further since application of (2) leads to $L_4 = -\frac{1}{16}L_1$ in this case. This expression then agrees with the one obtained by Deser and Lindström (1980), that is the current $D^{\lambda\mu}$ disappears.

3. Reduction

In order to show that I_{grav} reduces to the 4D invariant of O(2) supergravity given by Deser and Kay (1978), we write all spinor expressions in Dirac notation, and with the representation of the 6D Γ-matrices as defined by Brink *et al* (1977), the field strength $\frac{1}{2}(1+\Gamma_7)f^{\alpha\beta}_{\text{DIRAC}} \equiv f_+^{\alpha\beta}$ reduces to the only non-vanishing components

$$f_+{}^{ab} = \begin{bmatrix} f_+{}^{(4\text{D})ab} \\ f_-{}^{(4\text{D})ab} \end{bmatrix}$$

where $f_\pm{}^{(4\text{D})ab}$ are complex 4D fields related to the O(2) fields by

$$f_\pm{}^{(4\text{D})ab} = (1/\sqrt{2})(f_\pm^{1\,ab} + \mathrm{i} f_\pm^{2\,ab})$$

and f^i_{ab} $(i = 1, 2)$ are 4D Majorana spinors. In the same procedure $R_{\alpha\beta\gamma\delta}$ reduces to

$$R_{abcd} = R^{(4\text{D})}{}_{abcd}, \qquad R_{abc4} = (1/\sqrt{2})\,\partial_c{}^*F_{ab}{}^{(4\text{D})}, \qquad R_{abc5} = -(1/\sqrt{2})\,\partial_c F_{ab}{}^{(4\text{D})}.$$

The reduction of the fermionic part as well as the spin-1 part of I_{grav} is similar to the Yang–Mills case. The result shows that $I_{\text{grav}} = -48\Delta_3\mathscr{L}[\text{O}(2)]$, where $\Delta_3\mathscr{L}[\text{O}(2)]$ is the invariant found by Deser and Kay (1978).

We explicitly demonstrate here the reduction of the pure spin-2 part of I_{grav} which yields the square of the Bel–Robinson tensor. From (3) and (4) we learn that pure spin-2 contributions are contained only in L_2, where they arise from $T^{\mu\nu}{}_{\alpha\beta}$ and $D^{\mu\nu}{}_{\alpha\beta}$ terms. We denote this part of L_2 by $[L_2]_R$ and find that

$$\begin{aligned}[L_2]_R &= [-8(T+D)^2]_R \\ &= -8(T^{ab}{}_{cd}T_{ab}{}^{cd} + 2T^{44}{}_{cd}T^{44\,cd} + 2D^{45}{}_{cd}D^{45\,cd})\end{aligned} \tag{5}$$

where

$$\begin{aligned} T^{ab}{}_{cd} &= R_{(c}{}^{ef(a}R_{d)ef}{}^{b)} + \eta^{ab}T^{44}{}_{cd}, \\ T^{44}{}_{cd} &= -\tfrac{1}{2}R_{(c}{}^{efg}R_{d)efg}, \\ D^{45}{}_{cd} &= \tfrac{1}{2}R^*{}_{(c}{}^{efg}R_{d)efg}. \end{aligned}$$

At this point it is very convenient to introduce the curvature (Weyl) spinor χ_{ABCD} (Penrose 1960) which is totally symmetric on shell. In terms of χ_{ABCD} the various terms in (5) take the form

$$R_{(a}{}^{efg}R_{b)efg} = \tfrac{1}{2}x\eta_{ab}, \qquad R^*{}_{(a}{}^{efg}R_{b)efg} = \tfrac{1}{2}y\eta_{ab},$$
$$R_{(a}{}^{ef(c}R_{b)ef}{}^{d)} = \tfrac{1}{4}\eta_{ab}\eta^{cd}x + \tfrac{1}{128}\bar{\sigma}_{(a}{}^{\dot{A}A}\bar{\sigma}_{b)}{}^{\dot{B}B}\sigma^{(c}{}_{C\dot{C}}\sigma^{d)}{}_{D\dot{D}}\chi_{AB}{}^{CD}\bar{\chi}_{\dot{A}\dot{B}}{}^{\dot{C}\dot{D}},$$

where $x = \frac{1}{16}(\chi\chi + \bar{\chi}\bar{\chi})$ and $y = -\frac{1}{16}\mathrm{i}(\chi\chi - \bar{\chi}\bar{\chi})$.

The square of the Bel–Robinson tensor is therefore given by

$$T^2 \equiv \tfrac{1}{4}T^{ab}{}_{cd}T_{ab}{}^{cd} = \tfrac{1}{256}\chi\chi\bar{\chi}\bar{\chi}$$

and $[L_2]_R$ becomes

$$[L_2]_R = -48T^2.$$

We conclude that the gravitational contribution to the three-loop counterterm can be obtained from higher-dimensional representations after reduction, as well. In particular, the procedure given here provides a systematic way to determine all currents involved in the corresponding counterterms. An application to higher dimensions, especially to ten dimensions, seems to be straightforward, since the guiding Yang–Mills multiplet in this case has exactly the same form as in six and four dimensions (Gliozzi *et al* 1977).

In the eleven-dimensional case we have neither a Weyl representation nor a Yang–Mills multiplet. However, it should be possible to construct the corresponding on-shell superfield, which then allows the construction for a Noether current.

After this work was completed, I received a paper by Howe and Lindström (1980) in which they find all three-loop counterterms for O(N) supergravity, $N \leqslant 4$, and a seven-loop counterterm for O(8) supergravity using 4D on-shell superfields as found in the work by Brink and Howe (1979).

Acknowledgment

I would like to thank Dr S Deser for suggesting the problem and for helpful discussions.

References

Akyeampong D A and Delbourgo R 1973 *Nuovo Cimento* **17A** 573
Brink L and Howe P 1979 *Phys. Lett.* **88B** 268
Brink L, Schwarz J H and Scherk J 1977 *Nucl. Phys.* B **121** 77
Deser S and Kay J H 1978 *Phys. Lett.* **76B** 400
Dèser S and Lindström U 1980 *Phys. Lett.* **90B** 68
Ferrara S and Zumino B 1975 *Nucl. Phys.* B **87** 207
Gliozzi F, Scherk J and Olive D 1977 *Nucl. Phys.* B **100** 253
Howe P and Lindström U 1980 *Preprint University of Stockholm*
Penrose R 1960 *Ann. Phys.* **10** 171
Scherk J 1979 in *Gravitation: Recent Developments* (New York: Plenum)
Siegel W 1979 *Phys. Lett.* **80B** 220
Sohnius M F 1979 *Phys. Lett.* **81B** 8